Voltage → potential difference

V_{ab}

V_{ab}

node voltage ← wrt. to ground
V_a, V_b (g)
$\|$ $\|$ (ref. node)
V_{ag} V_{bg}

"art" ⟶ "algorithm"

CHAPTER 3

Methods of Analysis

Here we apply the fundamental laws of circuit theory (Ohm's Law & Kirchhoff's Laws) to develop two powerful techniques for circuit analysis.
1. Nodal Analysis (based on KCL)
2. Mesh Analysis (based on KVL)
This is the ==*most important* chapter for our course.==

3.1. Nodal Analysis

Here, we analyze circuit using **node voltages** as the circuit variables.

EXAMPLE 3.1.1. Back to Example 2.5.7

⓪ There are three nodes.

① use node b as the ref. node.

Ex. $V_a = 4 V$

means

$V_{ab} = 4 V$

①ˢ $V_c = 12 V$

$V_c = 12 V$

i 4 Ω a i_0 ④

12 V   6 Ω v_0 3 Ω

b

$i_0 = \dfrac{V_a - V_b}{3} = \dfrac{V_a - 0}{3}$

$= \dfrac{4}{3}$

KCL
② @ node a ← Make sure that you indicate where
the KCL is applied.   ↑ which node

$\dfrac{V_a - 12}{4} + \dfrac{V_a - 0}{6} + \dfrac{V_a - 0}{3} = 0$
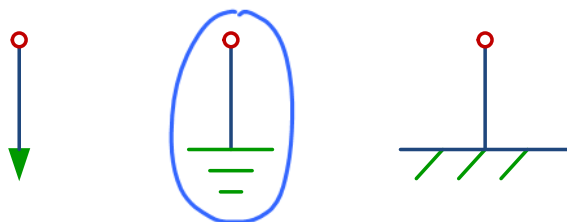
③ $V_a = 4 V$

3.1.2. **Steps to Determine Node Voltages:**

Step 0: Determine the number of nodes n.

Step 1: Select a node as a reference node (ground node). Assign voltages v_1, v_2, $\cdots$, v_{n-1} to the remaining $n-1$ nodes.

- The voltage are now referenced with respect to the reference node.
- The ground node is assumed to have 0 potential.

- Recall that voltages are measured between two points. In this case, the second point is always the ground node and hence we agree not to talk about it but know that all these node voltages are in fact the values of the voltages with respect to the reference node.
- If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source.

Step 2: Apply KCL to each of the ~~$n-1$~~ *remaining* nonreference nodes.

  (a) Use Ohm's law to express the branch currents in terms of node voltages.

If you are lazy, remember this

$$I = \frac{V_a - V_b}{R} \qquad I = \frac{V_b - V_a}{R}$$
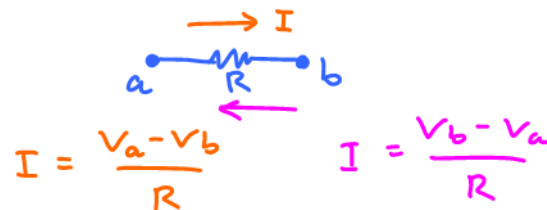
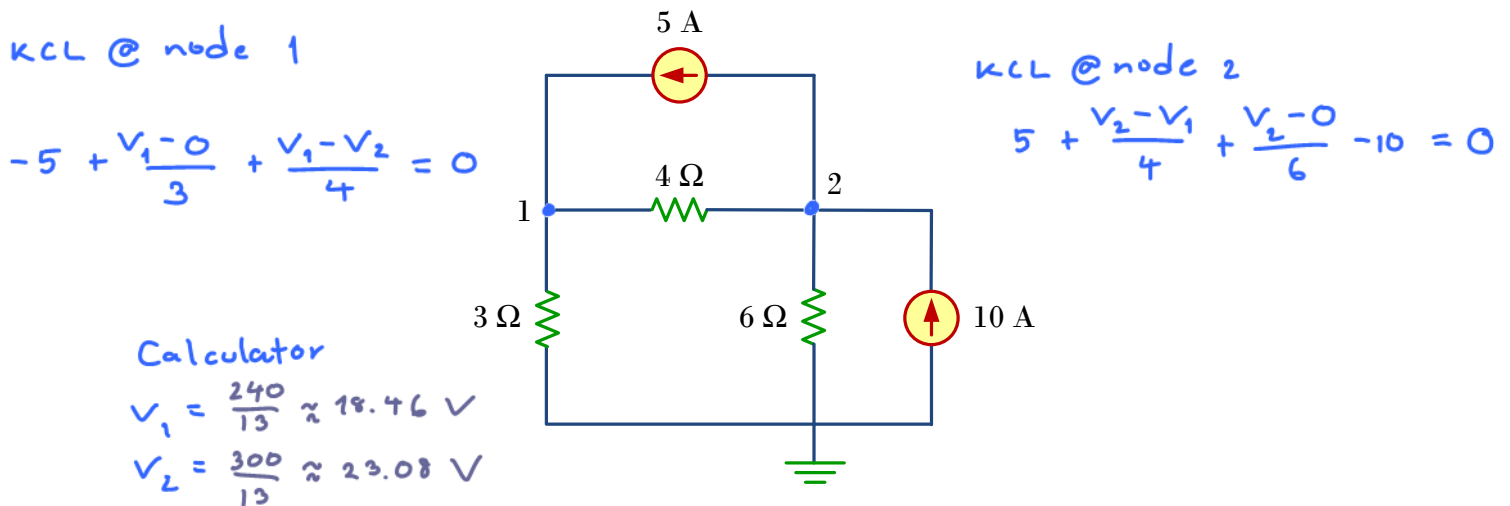 (b) Current source automatically gives current value. Caution: Watch out for the direction of the arrow.

Step 3: Solve the resulting *simultaneous equations* to obtain the unknown node voltages.

3.1.3. Remarks: There are multiple methods to solve the simultaneous equations in Step 3.

- Method 1: Elimination technique (good for a few variables)
- Method 2: Write in term of matrix and vectors (write $Ax = b$), then use Cramer's rule.
- Method 3: Use
    - computer (e.g., MATLAB) to find A^{-1} and then find $x = A^{-1}b$
    - calculator (fx-991MS can solve simultaneous linear equations with two or three unknowns.)

EXAMPLE 3.1.4. Calculate the node voltages in the circuit below.

KCL @ node 1

$$-5 + \frac{V_1 - 0}{3} + \frac{V_1 - V_2}{4} = 0$$

5 A

4 Ω

3 Ω   6 Ω   10 A

2

1

KCL @ node 2

$$5 + \frac{V_2 - V_1}{4} + \frac{V_2 - 0}{6} - 10 = 0$$

Calculator

$$V_1 = \frac{240}{13} \approx 18.46 \text{ V}$$

$$V_2 = \frac{300}{13} \approx 23.08 \text{ V}$$

EXAMPLE 3.1.5. Calculate the node voltages in the circuit below.

KCL @ node 1

$$\frac{V_1-V_3}{4} + \frac{V_1-V_2}{2} - 3 = 0$$

KCL @ node 2

$$\frac{V_2-V_1}{2} + \frac{V_2-0}{4} + \frac{V_2-V_3}{8} = 0$$

KCL @ node 3

$$\frac{V_3-V_1}{4} + \frac{V_3-V_2}{8} + 2i_x = 0$$

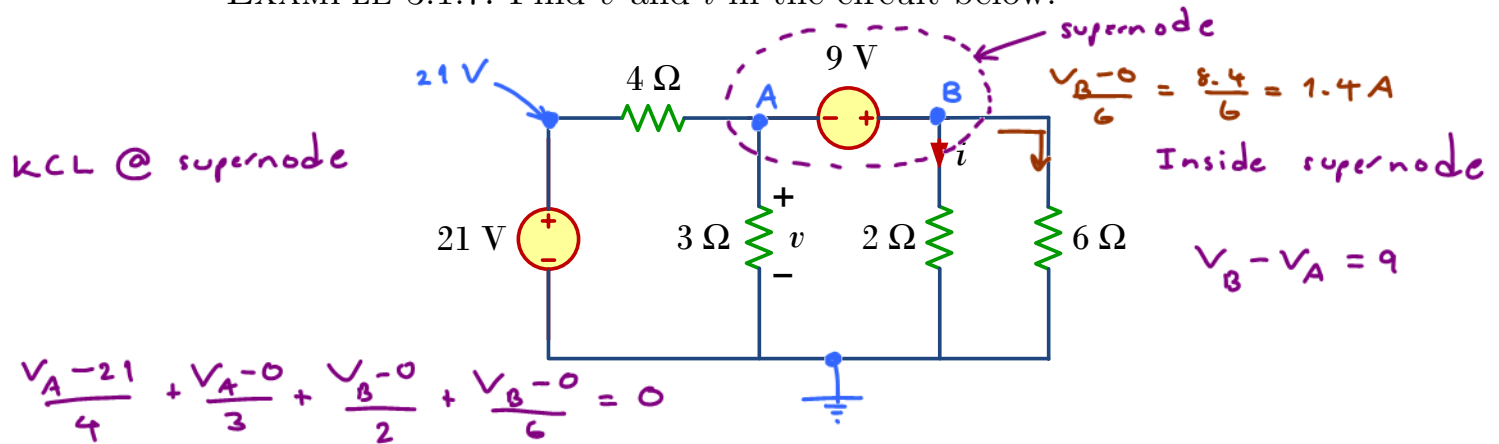$$\frac{V_3-V_1}{4} + \frac{V_3-V_2}{8} + 2\left(\frac{V_1-V_2}{2}\right) = 0$$

$$i_x = \frac{V_1-V_2}{2}$$

$$V_1 = \frac{24}{5} = 4.8 \text{ V}$$

$$V_2 = \frac{12}{5} = 2.4 \text{ V}$$

$$V_3 = -\frac{12}{5} = -2.4 \text{ V}$$

3.1.6. **Special Case**: If there is a voltage source connected between two nonreference nodes, the two nonreference nodes form a **supernode**. We apply both KCL and KVL to determine the node voltages.

EXAMPLE 3.1.7. Find v and i in the circuit below.

KCL @ supernode

$$\frac{V_A - 21}{4} + \frac{V_A - 0}{3} + \frac{V_B - 0}{2} + \frac{V_B - 0}{6} = 0$$

supernode

$$\frac{V_B - 0}{6} = \frac{8.4}{6} = 1.4 A$$

Inside supernode

$$V_B - V_A = 9$$

Calculator:

$$V_A = -0.6 \, V \qquad\qquad v = V_A - 0 = -0.6 \, V$$

$$V_B = 8.4 \, V \qquad\qquad i = \frac{V_B - 0}{2} = \frac{8.4}{2} = 4.2 \, A$$

3.1.8. Note the following properties of a supernode:
(a) The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages.
(b) A supernode has no voltage of its own.
(c) We can have more than two nodes forming a single supernode.
(d) The supernodes are treated differently because nodal analysis requires knowing the current through each element. However, there is no way of knowing the current through a voltage source in advance.

3.2. Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using **mesh currents** as the circuit variables.

DEFINITION 3.2.1. **Mesh** is a loop which does not contain any other loop within it.

3.2.2. **Steps to Determine Mesh Currents:**

Step 0: Determine the number of meshes n.

**Step 1:** Assign mesh currents[1] i_1, i_2, ..., i_n, to the n meshes.
- The direction of the mesh current is arbitrary–(clockwise or counterclockwise)–and does not affect the validity of the solution.
- For convenience, we define currents flow in the clockwise (CW) direction.

Step 2a From the current direction in each mesh, define the voltage drop polarities.

2b **Step 3:** Apply KVL to each of the n meshes.

Use Ohm's law to express the voltages in terms of the mesh current.
- Tip (for combining Step 2a and 2b: Go around the loop in the same direction as the mesh current (of that mesh). (Note that according to our agreement above, this is in the CW direction.) When we pass a resistor R, the voltage drops by $I \times R$ where I is the branch current (algebraic sum of mesh currents) through that resistor in the CW direction.

*a R b Going from a to b,
the voltage drops by IR.
I sum of mesh currents (in the mesh direction; if not, "−")*
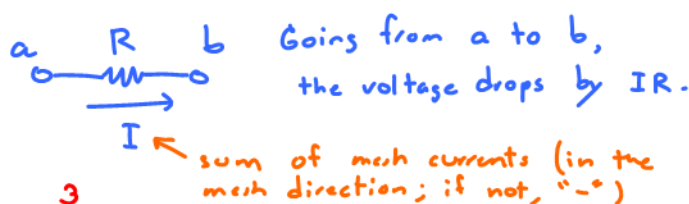
Step 4: Solve the resulting n simultaneous equations to get the mesh currents.

Step 4: Anything else in the circuit can be found using the mesh currents.

	Nodal Analysis	Mesh Analysis
variables	node voltages $V_A, V_B, \ldots$	mesh currents $i_1, i_2, \ldots$
underlying principles	KCL + Ohm's law	KVL + Ohm's law

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[1]Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously.

EXAMPLE 3.2.3. Back to Example 2.5.7

Mesh 1 : KVL

$+12 - V_1 - V_2 = 0$

$+12 - i_1 \times 4$

$-(i_1 - i_2) \times 6 = 0$

12 V

i_1

Calculator : $i_1 = 2 A$

$i_2 = \frac{4}{3} A$

4 Ω

a

i_0

$-V_3$ i_2 v_0 $+V_4$

6 Ω   3 Ω

b

Mesh 2 : KVL

$-V_3 - V_4 = 0$

$-(i_2 - i_1)6 - i_2 \times 3 = 0$

$i = i_1 = 2 A$

$i_0 = i_2 = \frac{4}{3} A$

$v_0 = i_2 \times 3 = \frac{4}{3} \times 3 = 4 V$

↑

Ohm's law

EXAMPLE 3.2.4. Find the branch currents $I_1, I_2,$ and I_3 using mesh analysis.

Mesh 1: KVL

$15 - i_1 \times 5 - (i_1 - i_2) \times 10 - 10 = 0$

Mesh 2 : KVL

$10 - (i_2 - i_1)10 - i_2 \times 6 - i_2 \times 4 = 0$

Calculator : $i_1 = 1 A$

$i_2 = 1 A$

$+ V_{AB} -$

I_1 5 Ω   A I_2 6 Ω   B

D $+ 5V -$

I_3

15 V i_1 i_2 10 Ω

10 Ω

4 Ω

10 V

C

$I_1 = i_1 = 1 A$

$I_2 = i_2 = 1 A$

$I_3 = i_1 - i_2 = 0 A$

$V_{AB} = I_2 \times 6 = 1 \times 6 = 6 V$

$V_{CA} = V_C - V_A = -10 V$

$V_C + 10 = V_A$

$V_C + 15 - 5 = V_A$

3.3. Remarks on Nodal Analysis and Mesh Analysis

3.3.1. Nodal analysis applies KCL to find unknown (node) voltages in a given circuit, while mesh analysis applies KVL to find unknown (mesh) currents.

3.3.2. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is *planar*.

- A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is nonplanar.

3.3.3. **Nodal Analysis vs. Mesh Analysis**: Given a network to be analyzed, how do we know which method is better or more efficient?

Suggestion: You should be familiar with both methods. Choose the method that results in smaller number of variables or equations.

- A circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis.

You can also use one method to check your results of the other method.

3.3.4. Nodal analysis and mesh analysis can also be used to find equivalent resistance of a part of a circuit.

This becomes extremely useful when the techniques that we studied in the previous chapter cannot be directly applied (, e.g., when we can't find resistors that are in parallel or in series; they are all connected in some "strange" configuration.)

The are two approaches to this kind of problems.

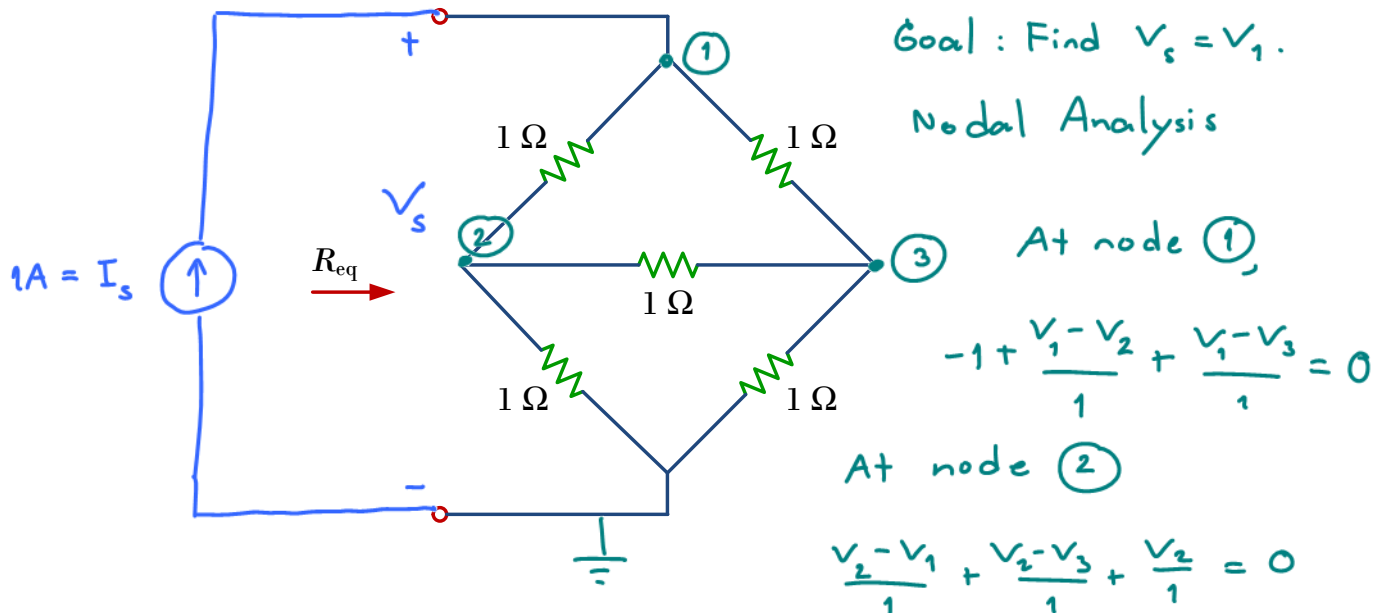(a) Apply 1 V voltage source across the terminals, find the corresponding current I through the voltage source. Then,

$$R_{eq} = \frac{V}{I} = \frac{1}{I}.$$

(b) Put 1 A current source through the terminals, find the corresponding voltage V across the current source. Then,

$$R_{eq} = \frac{V}{I} = \frac{V}{1} = V. \quad \leftarrow \text{we will try to use this to solve the next example.}$$

EXAMPLE 3.3.5. Find the equivalent resistance for the following circuit

Goal : Find $V_s = V_1$.

Nodal Analysis

At node ①,

$$-1 + \frac{V_1 - V_2}{1} + \frac{V_1 - V_3}{1} = 0$$

At node ②

$$\frac{V_2 - V_1}{1} + \frac{V_2 - V_3}{1} + \frac{V_2}{1} = 0$$

At node ③, $\frac{V_3 - V_1}{1} + \frac{V_3 - V_2}{1} + \frac{V_3}{1} = 0$

Calculator $\Rightarrow$ $V_1 = 1\,V$

$$R_{eq} = \frac{V_s}{I_s} = \frac{V_1}{1} = 1\,\Omega$$